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Special virtual fields for the direct determination of material parameters with the virtual fields method. 1—Principle and definition

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Abstract

This paper deals with the direct and simultaneous estimation of parameters used in some constitutive laws. Whole-field data captured in mechanical configurations which give rise to heterogeneous stress fields are processed. Since no analytical relationship is available between measured data and unknown parameters, a specific procedure based on a relevant use of the principle of virtual work is proposed. The main advantage is to provide directly the unknown parameters. The main features of the method are described in the paper.

Keywords: Anisotropy; Heterogeneous stress field; Identification; Inverse problem; Virtual fields method

1. Introduction

The determination of parameters that govern the constitutive equations of advanced materials is a challenge that becomes difficult to face when the number of parameters is significant. Such a case occurs when anisotropic materials or refined non-linear laws are considered. The usual approach consists of performing several tests like tensile tests and fitting the model with the experimental data. However, the number of tests increases with the number of parameters. Moreover, parasitic effects can disturb the stress field which is usually expected to be uniform when tensile/compressive tests are performed. This case occurs for instance when off-axis anisotropic materials are tested (Pagano and Halpin, 1968; Pindera and Herakovich, 1986; Pierron et al., 1998) or when the span-to-depth ratio of the coupon is small, as in the case of the through-thickness testing of laminated structures (Gipple and Hoyns, 1994; Mespoulet, 1998; Broughton et al., 1998; Broughton, 1994 for instance).

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These drawbacks can be avoided by using tests on plate specimens. In this case, only one coupon is tested and the stress field which occurs in the specimen is heterogeneous. Hence several parameters influence the mechanical response of the coupon. They can therefore be simultaneously identified if a suitable strategy is used. For instance, the natural frequencies of vibrating composite plates depend on the bending stiffnesses. To measure them, mixed experimental/numerical procedures have been proposed in the literature (Sol, 1986; Wilde, 1990; Pedersen and Frederiksen, 1992; Ayorinde and Gibson, 1993; Deobald and Gibson, 1988; Mota Soares et al., 1993; Araujo et al., 1996; Frederiksen, 1997; Cunha and Piranda, 1999; Bledzki et al., 1999; Hwang and Chang, 2001). First a numerical model is built up with the finite element or the Rayleigh-Ritz methods. This model provides the first natural frequencies which are used with corresponding measured ones to define a residual which is minimized iteratively with respect to the parameters to be identified. Such approaches are efficient but limited to elastic (or viscoelastic) *bending* properties: in-plane stiffness parameters cannot be identified with them. Moreover, since the strain level is very low in dynamics, non-linear responses of materials cannot be detected and identified. Another way is to identify the parameters from static tests on a plate specimen in which an heterogeneous stress field occurs. If a closed-form solution for the actual strain/stress field is available, some local measurements with strain gauges lead to the parameters (Prabhakaran and Chermahini, 1984). In this case however, only the elastic moduli of an orthotropic material can be measured and one has no freedom concerning the shape of the specimen. Moreover, the boundary conditions in the experiment must be exactly those of the model. When no closed-form solution is available, procedures based on the updating of finite element models have been developed (Hendricks, 1991; Rouger et al., 1990; Okada et al., 1999; Wang and Kam, 2001), either in the case of a reduced number of measurements or in the case of whole-field data. This last case occurs when the strain field is measured onto the surface of the specimen with an optical method. Updating finite element models exhibits however some drawbacks which will be discussed below. When whole-field data are processed, it will be shown that these drawbacks can be avoided by using another strategy called the virtual fields method (VFM). This method consists in applying the principle of virtual work to the specimen with particular virtual fields (Grédiac, 1989).

In the present paper, the objective is to propose a dramatic improvement of the VFM, since a procedure for automatically finding virtual fields is proposed. The headlines of the method are recalled in the first part of the paper. Then, it is shown that applying the VFM with some special virtual fields directly provides the unknown parameters independently one from the other. Some cases of non-linear mechanical responses are also considered. Finally, a general numerical procedure for finding in practice these special virtual fields is proposed. The numerical simulation of the approach is described in a companion paper (Grédiac et al., 2002).

2. Determining material constants with the virtual fields method

Let us consider a solid of any shape subjected to prescribed loading and displacement (see Fig. 1). In the general case, no closed-form solution for the actual displacement/strain field is available and the problem is to retrieve the parameters governing the constitutive equations of the material assuming that the displacement/strain fields as well as the loading are measured with some suitable devices. If body-forces distributions are neglected, we only have as load possibility traction (or stress vector) $\mathbf{T}(M,\mathbf{n})$ over S_f, where M is any point of S_f and $\mathbf{n}$ the vector perpendicular to S_f at point M. Over the remaining points of the boundary: S_u, the displacement field $\mathbf{u}$ is prescribed: $\mathbf{u} = \bar{\mathbf{u}}$. The principle of virtual work may be written as

$$\int_V \sigma : \epsilon^* \, \mathrm{d}V = \int_{S_f} \mathbf{T}(M,\mathbf{n}) \cdot \mathbf{u}^*(M) \, \mathrm{d}S \tag{1}$$

where σ is the stress field, ϵ^* the virtual strain field, V the volume of the solid, u^* the virtual displacement field. This equation is valid for any kinematically admissible (KA) virtual field $(\mathbf{u}^*, \epsilon^*)$, that is, $\mathbf{u}^* = \mathbf{0}$ over

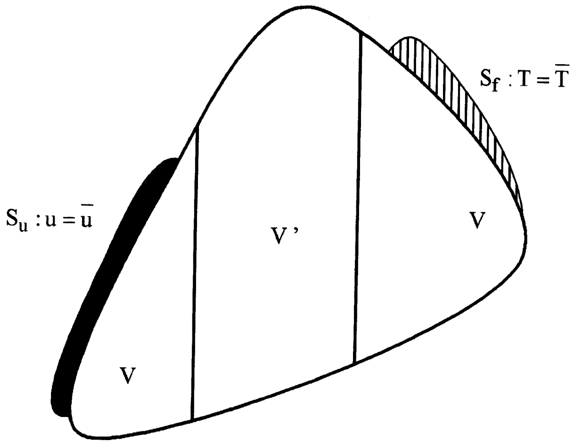

Fig. 1. Solid of any shape.

S_u. Let us now introduce the constitutive equations which are assumed to be polynomial. With the usual rules for contracted indices ($xx \to x$, $yy \to y$, $zz \to z$, $yz \to q$, $xz \to r$, $xy \to s$) and summation of these indices, the stress/strain relation may be written in the case of linear constitutive equations

$$\sigma_i = Q_{ij}\epsilon_j \tag{2}$$

Feeding the above constitutive equations in Eq. (1), the principle of virtual work may be written as

$$\int_V Q_{ij}\epsilon_j\epsilon_i^* \,\mathrm{d}V = \int_{S_f} \mathbf{T}(M,\mathbf{n}) \cdot \mathbf{u}^*(M)\,\mathrm{d}S \tag{3}$$

Assuming that the constitutive material is homogeneous, the Q_{ij}'s do not depend on x, y and z, thus

$$Q_{ij}\int_V \epsilon_j\epsilon_i^* \,\mathrm{d}V = \int_{S_f} \mathbf{T}(M,\mathbf{n}) \cdot \mathbf{u}^*(M)\,\mathrm{d}S \tag{4}$$

The problem is to determine the Q_{ij}'s. It is a trivial matter to see that any different virtual field provides a new linear equation of the type of Eq. (4). The so-called VFM consists in writing the above equation with as many different virtual fields as unknowns. This leads to a system of linear equations

$$\mathbf{PQ} = \mathbf{R} \tag{5}$$

where $\mathbf{P}$ is a square matrix and $\mathbf{Q}$ a vector whose components are the Q_{ij}'s. The above system provides the unknown parameters after inversion. This method, introduced first in Grédiac (1989), has been successfully applied to bending (either in statics (Grédiac and Vautrin, 1990; Grédiac, 1996a,b), or in dynamics (Grédiac and Paris, 1996; Grédiac et al., 1998)), to in-plane (Grédiac and Pierron, 1998; Grédiac et al., 1999) and more recently to through-thickness composite characterization, either with a linear elastic (Pierron et al., 2000; Pierron and Grédiac, 2000) or a non-linear response (Grédiac et al., 2001). A keypoint of this general identification method is the determination of the virtual fields, since these fields directly influence the degree of independence of the equations in the linear system (5). In these previous studies, the virtual fields were chosen intuitively, using a trial-and-error procedure, in such a way that the different equations of the system were "sufficiently" independent. The independence of the equations is in fact directly related to the sensitivity of the identified parameters to noisy measured displacement/strain components. Hence, it directly influences an important aspect of the applicability of the method: its stability. It is therefore essential to obtain virtual fields leading to independent equations when experimental data are processed since noise is unavoidable in this case. Some practical rules were applied for finding these fields in practice. For instance, the fields were built up in such a way that the equations were partially uncoupled, i.e. some P_{ij}'s were zero in matrix **P**. These fields led however to rather satisfactory results but they were not demonstrated to be the optimal ones. In particular, it was not possible to find by hand a set of independent virtual fields that provided directly the unknown material parameters, since the unknowns remained coupled in the linear equation provided by the principle of virtual work, and then in the linear system (5). This system had therefore to be inverted without any guarantee that the independence of the equations was "sufficient" in all cases.

The main improvement of the present work is to provide a procedure which allows the determination of virtual fields which directly lead to uncoupled equations in system (5). The VFM becomes therefore much more attractive since no intuitive guess of the virtual fields is needed. Moreover, because of the uncoupled equations, the stability of the procedure is expected to be better. Another feature shown in a companion paper (Grédiac et al., 2002) is the fact that an infinite number of virtual fields is available for each unknown parameter. Among all these virtual fields, optimized fields with respect to noisy data will be found. Such optimized fields will lead to optimized values for the unknown parameters.

3. Special virtual fields for the direct identification of the unknown constants

3.1. Controlling the coefficients of the unknowns in the principle of virtual work

Let us take into account the symmetry of the stiffness matrix ($Q_{ij} = Q_{ji}$) in Eq. (4). Then the non-diagonal stiffness components may be factorized. With this factorization, the synthetic expression of the coefficient of any stiffness component Q_{ij} in the principle of virtual work ($i = j$ or $i \neq j$) may be written as $(1/(1+\delta_{ij}))\int_V \left(\epsilon_j\epsilon_i^* + \epsilon_i\epsilon_j^*\right)\mathrm{d}V$, where δ_{ij} is the Kronecker delta symbol. If a material constant (say Q_{pq} for instance, $p \neq q$) is to be identified, it is simply to be written that its coefficient is unity in the principle of virtual work whereas the coefficients of the other material parameters are set to zero

$$Q_{xx}\underbrace{\int_V \epsilon_x\epsilon_x^*\,\mathrm{d}V}_{=0m^3} + \cdots + Q_{ij}\underbrace{\frac{1}{1+\delta_{ij}}\int_V(\epsilon_j\epsilon_i^* + \epsilon_i\epsilon_j^*)\,\mathrm{d}V}_{=0m^3} + \cdots + Q_{pq}\underbrace{\int_V(\epsilon_q\epsilon_p^* + \epsilon_p\epsilon_q^*)\,\mathrm{d}V}_{=1m^3} + \cdots + Q_{ss}\underbrace{\int_V \epsilon_s\epsilon_s^*\,\mathrm{d}V}_{=0m^3}$$
$$= \int_{S_f} \mathbf{T}(M,\mathbf{n})\cdot\mathbf{u}^*(M)\,\mathrm{d}S \tag{6}$$

To obtain such a set of particular coefficients in Eq. (6), at least one so-called *special virtual field* is to be found. This virtual field, denoted hereafter $\hat{\mathbf{u}}^*$, obeys the following conditions

$$\begin{cases} \text{condition 1: } \hat{\mathbf{u}}^* \text{ is Kinematically Admissible (K.A.)} \\ \text{condition 2: } \begin{cases} \frac{1}{1+\delta_{ij}}\int_V(\epsilon_j\hat{\epsilon}_i^* + \epsilon_i\hat{\epsilon}_j^*)\,\mathrm{d}V = 0 & \forall i \neq p \text{ or } j \neq q \\ \frac{1}{1+\delta_{ij}}\int_V(\epsilon_j\hat{\epsilon}_i^* + \epsilon_i\hat{\epsilon}_j^*)\,\mathrm{d}V = 1 & \text{if } i = p \text{ and } j = q \end{cases} \end{cases} \tag{7}$$

where $\hat{\epsilon}^*$ is the special virtual strain field derived from the special virtual displacement field $\hat{\mathbf{u}}^*$. It is equivalent to saying that a set of virtual fields has to be found such that matrix **P** in system (5) becomes unity: $\mathbf{P} = \mathbf{I}$.

3.2. Additional conditions

3.2.1. Applied load

In practice, the distribution of the load over S_f remains unknown. Only the projection P of the resulting force along a given direction **t** can be measured, typically by the load cell of a testing machine. P is defined by

$$P = \mathbf{t} \cdot \int_{S_f} \mathbf{T}(M, \mathbf{n})\, \mathrm{d}S \tag{8}$$

where the "." operator indicates the scalar product between two vectors.

Consequently, the special virtual fields must be such that

$$\text{condition 3:} \;\; \hat{\mathbf{u}}^*(M) = \alpha \mathbf{t} \;\; \forall M \in S_f \tag{9}$$

where α is any non-zero constant. Indeed, with such a virtual field, the virtual work of the applied load (i.e. the right-hand side part of Eq. (1)) can be expressed as a function of the measured resulting force P along direction $\mathbf{t}$

$$\int_{S_f} \mathbf{T}(M, \mathbf{n}) \cdot \hat{\mathbf{u}}^*(M)\, \mathrm{d}S = \int_{S_f} \mathbf{T}(M, \mathbf{n}) \cdot \alpha \mathbf{t}\, \mathrm{d}S = \alpha \mathbf{t} \cdot \int_{S_f} \mathbf{T}(M, \mathbf{n})\, \mathrm{d}S = \alpha P \tag{10}$$

In any other case, the distribution of the load over S_f must be known with measurements or through an assumption to calculate the right-hand side in Eq. (1).

3.2.2. Actual field known over V' only

In practice, another conditions must be verified if the actual displacement/strain field is only known in a part V' of V. Let S' be the boundary between V and V'. Since the actual field is unknown over $V - V'$, its contribution to the integrals in condition 2 must be eliminated. Then, the special virtual field $\hat{\mathbf{u}}^*$ is such that

$$\hat{\epsilon}_i^*(M) = 0 \quad \forall M \in V - V' \tag{11}$$

in other words, $\hat{\mathbf{u}}^*$ is solid-rigid like over $V - V'$. Finally, the special virtual displacement field must be continuous over the whole specimen, especially at the boundary S' between V and V'. Thus, we can write

if the actual field $(\mathbf{u}, \epsilon)$ *is known over* $V' \in V$ *only, then*

$$\begin{cases} \text{condition 4: } \hat{\mathbf{u}}^* \text{ is solid-rigid like over } V - V' \\ \text{condition 5: } \hat{\mathbf{u}}^* \text{ is continuous over } S' \end{cases} \tag{12}$$

3.3. Conclusion

With such a special field $\hat{\mathbf{u}}^*$, we have the following expression for Q_{pq}

$$Q_{pq} = \int_{S_f} \mathbf{T}(M, \mathbf{n}) \cdot \hat{\mathbf{u}}^*(M)\, \mathrm{d}S = \alpha P \tag{13}$$

In this equation, Q_{pq} is directly equal to the virtual work of the external loading produced by the special virtual field $\hat{\mathbf{u}}^*$. This clearly shows that the VFM used with these special virtual fields directly extracts the unknown material parameters from the measured data which are the actual strain field ϵ over V or V' and the resulting force P. The result in Eq. (13) could be troublesome in terms of units since a virtual work is normally expressed in $N \times m$, but it must be emphasized that the unit of the integrals in Eq. (6), which are either equal to 0 or to 1, is in fact m^3. Hence, the unit of the right-hand side in Eq. (6) is $(N \times m)/m^3 = N/m^2 = Pa$.

It is difficult to discuss rigorously the existence of the special virtual field $\hat{\mathbf{u}}^*$ in the general case and for all the Q_{ij}'s. It is clear however that if the state of stress is uniform, at most six coefficients or six combinations of coefficients can be determined. In this case, the σ_i's are constant and Eq. (1) becomes

$$\sigma_i \int_V \epsilon_i^*\, \mathrm{d}V = \int_{S_f} T_i u_i^*\, \mathrm{d}S \tag{14}$$

It is clear that whatever the choice of the virtual field, six constant σ_i's and therefore six groups of mechanical parameters at most can be identified. In the case of plane stresses, three constant σ_i's and therefore three groups of mechanical parameters at most can be identified, while four are to be found for instance in the case of an orthotropic elastic law. As a conclusion, only heterogeneous stress/strain fields will be processed in the following to avoid this drawback.

3.4. Comparison with updating finite element models

Updating finite element model has been proposed in the literature to determine elastic constants of composite plates from measured natural frequencies (Ayorinde and Gibson, 1993; Deobald and Gibson, 1988; Mota Soares et al., 1993; Araujo et al., 1996; Frederiksen, 1997; Cunha and Piranda, 1999) or from displacement fields (Hendricks, 1991). Processing displacement data is also proposed in Okada et al. (1999) to find parameters governing the elastic/plastic relationship of a metal. It is therefore relevant to compare the present approach with the procedures based on the updating of finite element models.

The headlines of both the finite element method and the present VFM are recalled in Fig. 2. Two classical problems of the mechanics of solids are presented in this figure in the case of linear elasticity. In the first one, often referred to as the *direct problem*, the displacement, strain and stress fields are unknown and the material parameters are known. The second one is the problem presently under consideration. It is often referred to as the *inverse problem*: the displacement/strain field is known whereas the material parameters are unknown. It clearly appears that the finite element method is well suited for solving the first problem when no closed-form solution is available, since the displacement at the nodes **U** are directly determined by inverting $\mathbf{KU} = \mathbf{F}$. Updating a finite element model with respect to the material parameters for solving the inverse problem suffers however some drawbacks:

- Iterative calculations must be performed. At each step, the system $\mathbf{KU} = \mathbf{F}$ must be inverted and this can be time consuming. On the other hand, the VFM is a direct method.
- In practice, the load distribution remains generally unknown and only the global resulting force is measured. If a finite element calculation is performed, the load distribution must be input in the model. Hence these calculations can only be performed under some assumptions concerning this distribution. On the other hand, with the VFM, it will be possible to build special virtual fields that lead to a virtual work of the external loading which only depends on the resulting force (see condition 3 above), as illustrated in Grédiac et al. (2002) for instance.
- In the same way, the actual imposed displacement distribution over S_u is not necessarily null and can be unknown over some parts of S_u. This imposed distribution over S_u must be input in a finite element model. This can only be made with some assumptions. This problem is solved with the VFM since the virtual fields are KA.
- In most cases, initial values of the parameters must be guessed to initiate the iterations and these initial values play an important role in the convergence of the procedure.
- If non-linear constitutive equations are considered, iterative calculations must be performed for each set of material parameters. In this case, fitting experimental and numerical data would lead to very heavy calculations.

Eventually, the finite element method is the natural tool for solving problem 1 when no closed-form solution is available. Updating a finite element model with respect to the material constants is suitable when natural frequencies are considered as input data, but such a method is not well suited for solving problem 2, i.e. when displacement/strain fields are to be processed.

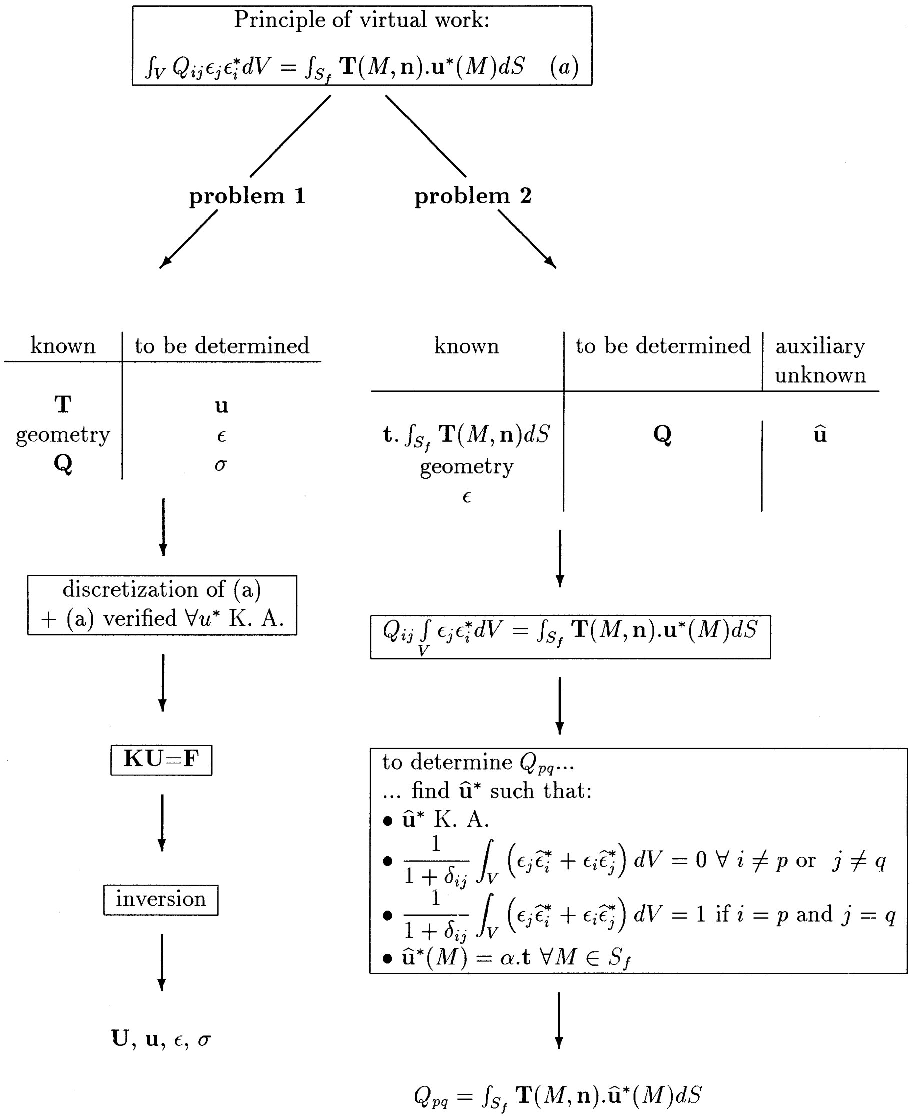

Fig. 2. The finite element method, as a solution of problem 1 and the VFM as a solution of problem 2 (case of elasticity).

On the other hand, it is worth noting that the VFM *directly* provides the unknown material parameters, since the Q_{ij}'s are directly expressed as functions of the measured field ($\mathbf{u}, \epsilon$) in Eq. (13). It clearly appears that *all* the measured data can be taken into account: the integrals in the principle of virtual work are obviously discretized, but the refinement can be much higher than in the finite element method since no inversion is required. Bearing in mind that an *infinite* number of admissible virtual fields verify the principle of virtual work, the virtual fields can be selected to solve the problem of the lack of experimental information on the actual loading distribution, as will be shown in the examples below. It is also worth noting that no assumption is made on the displacement/strain/stress fields, apart from the type of constitutive equations between the stress and strain components.

3.5. Functionally graded materials

Let us consider a material whose properties spatially change inside the solid. Some examples of such materials are described in Okada et al. (1999). The Q_{ij}'s now depend on x, y and z. For the sake of legibility and simplicity, let us assume that the evolution of each stiffness can be described by a polynomial of degree n of y only (Okada et al., 1999), we have

$$Q_{ij} = a_{ij_k} y^k \tag{15}$$

where the a_{ij_k}'s are constant and where k lies between 0 and a given value n. In this case, the principle of virtual work may be written as

$$\int_V (a_{ij_k} y^k)\epsilon_j \epsilon_i^* \,\mathrm{d}V = \int_{S_f} \mathbf{T}(M,\mathbf{n}) \cdot \mathbf{u}^*(M)\,\mathrm{d}S \tag{16}$$

Introducing the symmetry of the stiffness matrix, the above equation can be rewritten

$$\begin{aligned} &a_{xx0} \underbrace{\int_V \epsilon_x \epsilon_x^* \,\mathrm{d}V}_{=0m^3} + \cdots + a_{xx_n} \underbrace{\int_V y^x \epsilon_x \epsilon_x^* \,\mathrm{d}V}_{=0m^3} + \cdots + a_{ij_k} \underbrace{\frac{1}{1+\delta_{ij}} \int_V y^k \left(\epsilon_j \epsilon_i^* + \epsilon_i \epsilon_j^*\right) \mathrm{d}V}_{=0m^3} + \cdots \\ &+ a_{pq_r} \underbrace{\frac{1}{1+\delta_{pq}} \int_V y^r \left(\epsilon_q \epsilon_p^* + \epsilon_p \epsilon_q^*\right) \mathrm{d}V}_{=1m^3} + \cdots\cdots + a_{ss_n} \underbrace{\int_V y^n \epsilon_s \epsilon_s^* \,\mathrm{d}V}_{=0m^3} \\ &= \int_{S_f} \mathbf{T}(M,\mathbf{n}) \cdot \mathbf{u}^*(M)\,\mathrm{d}S \end{aligned} \tag{17}$$

The principle of virtual work can be written as a linear function of the a_{ij_k}'s to be determined. If we want to determine a given coefficient a_{pq_r}, we must find a special virtual field $\hat{\mathbf{u}}^*$ such that

$$\begin{cases} \text{condition 1: } \hat{\mathbf{u}}^* \text{ is K.A.} \\ \text{condition 2: } \begin{cases} \frac{1}{1+\delta_{ij}} \int_V y^k (\epsilon_j \epsilon_i^* + \epsilon_i \epsilon_j^*)\,\mathrm{d}V = 0 & \forall i \neq p \text{ or } j \neq q \text{ or } k \neq r \\ \frac{1}{1+\delta_{ij}} \int_V y^k (\epsilon_j \epsilon_i^* + \epsilon_i \epsilon_j^*)\,\mathrm{d}V = 1 & \text{if } i = p \text{ and } j = q \text{ and } k = r \end{cases} \end{cases} \tag{18}$$

With such a special virtual field $\hat{u}^*$, it follows

$$a_{pq_r} = \int_{S_f} \mathbf{T}(M,\mathbf{n}) \cdot \hat{\mathbf{u}}^*(M)\,\mathrm{d}S \tag{19}$$

If only the resulting force applied on the specimen is known, the above condition 3 must be verified. In the same way, in the particular case where the actual field is known only over $V' \in V$, the above conditions 4 and 5 must also be verified. Eventually, the special virtual field $\hat{\mathbf{u}}^*$ directly leads to the unknown a_{pq_r}. It should be pointed out that such an identification procedure could be used in quality assessment, for instance when the constitutive material is a polymer which is expected to exhibit a polymerization gradient.

3.6. Displacement fields as input data

As can be stated in Eq. (6), the actual strain field is involved in the coefficients of the unknown parameters. In practice however, the whole-field optical methods generally provide displacement fields and not strain fields.

Even though efficient algorithms are available to differentiate those displacement fields to get the strain fields (Surrel, 1994), it is clear that unavoidable noisy data disturb the displacement field, leading to some inaccuracy in the strain field. Hence, it should be pointed out that the strain components in the integrals can be transformed into displacements thanks to the Green's formula. For instance, using the notation with two indices, any coefficient of the mechanical parameter in Eq. (6) involves integrals of the type

$$\int_V u_{i,j} u^*_{k,l} \, \mathrm{d}V = -\int_V u_i u^*_{k,lj} \, \mathrm{d}V + \oint_{\partial V} u_i u^*_{k,l} \cos(\mathbf{n}, \mathbf{x_j}) \, \mathrm{d}S \tag{20}$$

It is clear that only the actual *displacement* field is considered in the right-hand side of the above equation (and not the actual *strain* field), but the final result is obtained by subtracting two quantities whose absolute values could be close. As a consequence, specific numerical simulations are to be performed to compare the stability of the two following approaches when noisy data are considered:

1. Differentiating the measured actual displacement field and computing the coefficients of the unknown parameters with the left-hand side in Eq. (20).
2. Computing the coefficients of the unknown parameters by subtracting the two terms in the right-hand side in Eq. (20) where the measured actual displacement field is involved.

3.7. Case of non-linear constitutive equations

3.7.1. Polynomial stress/strain relation

Let us consider the case where the stress/strain relation is a polynomial

$$\sigma_i = \sum_{t=1}^{n} Q_{ij}^{(t)} \epsilon_j^t \tag{21}$$

t is a power and (t) is a superscript.

Assuming that the $Q_{ij}^{(t)}$'s are constant over V, the principle of virtual work may be written as

$$\sum_{t=1}^{n} Q_{ij}^{(t)} \int_V \epsilon_j^t \epsilon_i^* \, \mathrm{d}V = \int_{S_f} \mathbf{T}(M, \mathbf{n}) \cdot \mathbf{u}^*(M) \, \mathrm{d}S \tag{22}$$

If we want to determine a material parameter $Q_{pq}^{(r)}$, we must find a special virtual field $\hat{\mathbf{u}}^*$ such that

$$\begin{cases} \text{condition 1: } \hat{\mathbf{u}}^* \text{ is K.A.} \\ \text{condition 2: } \begin{cases} \frac{1}{1+\delta_{ij}} \int_V (\epsilon_j^k \epsilon_i^* + \epsilon_i^k \epsilon_j^*) \, \mathrm{d}V = 0 & \forall i \neq p \text{ or } j \neq q \text{ or } k \neq r \\ \frac{1}{1+\delta_{ij}} \int_V (\epsilon_j^k \epsilon_i^* + \epsilon_i^k \epsilon_j^*) \, \mathrm{d}V = 1 & \text{if } \; i = p \text{ and } j = q \text{ and } k = r \end{cases} \end{cases} \tag{23}$$

With such a special virtual field denoted $\hat{\mathbf{u}}^*$, it follows

$$Q_{pq}^{(r)} = \int_{S_f} \mathbf{T}(M, \mathbf{n}) \cdot \hat{\mathbf{u}}^*(M) \, \mathrm{d}S \tag{24}$$

If only the resulting applied forces are known, conditions 3 in Section 3.2.1 must be verified. In the particular case where the actual field is known only over $V' \in V$, conditions 4 and 5 in Section 3.2.2 must also be verified. Eventually, the special virtual field $\hat{\mathbf{u}}^*$ directly leads to the unknown $Q_{pq}^{(r)}$.

3.7.2. Other cases

When the stress/strain relationship does not write as a polynomial, the principle of virtual work no more leads to a linear equation where the material parameters are the unknowns, but to a non-linear equation. In this case, for any virtual displacement field $\hat{\mathbf{u}}^*$, one can write a residual $R(\mathbf{u}^*)$ defined by

$$R(\mathbf{u}^*) = \left(\int_V \sigma_i \epsilon_i^* \, dV - \int_{S_f} T_i u_i^* \, dS \right)^2 \tag{25}$$

Introducing the non-linear constitutive equations $\sigma_i = f(\epsilon_x, \dots, \epsilon_s)$, where some material parameters are unknown, Eq. (25) becomes

$$R(\mathbf{u}^*) = \left(\int_V f(\epsilon_x, \dots, \epsilon_s) \epsilon_i^* \, dV - \int_{S_f} T_i u_i^* \, dS \right)^2 \tag{26}$$

This residual is zero for any virtual field when the tested specimen is in equilibrium. The idea for finding the unknown material parameters is first to build up a cost-function F with a limited number of residuals that correspond to N different virtual fields $\mathbf{u}^{*(i)}$, $i = 1, \dots, N$ chosen a priori and second to minimize this cost-function with respect to the unknown parameters

$$F = \sum_{i=1}^{N} R(\mathbf{u}^{*(i)}) \tag{27}$$

3.8. Conclusion

It has been shown that when the constitutive equations may be written as polynomials, some special virtual fields $\hat{\mathbf{u}}^*$, when they exist, directly provide the coefficients of these polynomials. Let us now examine the practical determination of these special virtual fields. For the sake of simplicity, only the case of linear anisotropic elasticity is addressed in the following section.

4. Practical determination of the special virtual fields

4.1. Choosing a basis for expressing the special virtual fields

Let us now examine the automatic determination of the special virtual fields $\hat{\mathbf{u}}^*$ with a numerical method. This issue is essential since the VFM becomes very general if such a procedure is available.

First, a basis of independent functions must be chosen to expand the virtual fields. Two main approaches can be investigated at this stage:

- The virtual displacement fields are defined piecewise, like the actual displacement field in the finite element method. In this case, the unknowns are the nodal virtual displacements and the virtual field is expanded with a local basis of suitable functions.
- The virtual displacement fields are expanded with the same basis of functions over the whole geometry. The unknowns are the coefficients of these functions.

Since one can expect more difficulties in the numerical implementation of the procedure for finding the virtual nodal displacements, only the second procedure is described herein. Any set of independent functions like polynomials or sine functions could be used to expand the virtual fields in this second procedure. The virtual fields to be determined are expressed as a weighted sum of these basis functions. It is difficult to predict a priori which type of function is the most suitable one. Indeed, it must be pointed out that the integrals in Eq. (6) involve both the actual and the virtual fields. The former one comes from a set of measurements known at a large number of points and is noisy in practice. The latter one, after expansion, involves the functions of the basis which are rigorously known at every point where the measurements are performed. From a numerical point of view, the present problem is therefore somewhat different of the well-known problems encountered in the finite element method since the functions which are computed in practice in this last case are known exactly. Polynomials have been arbitrarily chosen in the present work as basis functions to expand the virtual fields but it is clear that further studies must be undertaken to check which type of basis function leads to the best results. With polynomials, the components of the special virtual displacement field $\hat{\mathbf{u}}^*$ may be written as

$$\begin{cases} \hat{u}_x^* = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{o} A_{ijk}\left(\frac{x}{L}\right)^i\left(\frac{y}{H}\right)^j\left(\frac{z}{K}\right)^k \\ \hat{u}_y^* = \sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r} B_{ijk}\left(\frac{x}{L}\right)^i\left(\frac{y}{H}\right)^j\left(\frac{z}{K}\right)^k \\ \hat{u}_z^* = \sum_{i=0}^{s}\sum_{j=0}^{t}\sum_{k=0}^{u} C_{ijk}\left(\frac{x}{L}\right)^i\left(\frac{y}{H}\right)^j\left(\frac{z}{K}\right)^k \end{cases} \tag{28}$$

where L, H and K are representative dimensions along the x-, y- and z-directions respectively. This normalization provides in practice coefficients A_{ijk}, B_{ijk} and C_{ijk} which have about the same order of magnitude. This feature is useful for their numerical determination, as shown in Grédiac et al. (2002). The problem is now to determine the A_{ijk}'s, the B_{ijk}'s and the C_{ijk}'s from the three conditions listed above.

4.2. Equations induced by condition 1

As recalled in condition 1, the virtual displacement field must be first KA. Two cases may occur in practice.

4.2.1. The solid is supported at a finite number of points

If we assume that S_u reduces to a finite set of points M_v, $v = 1, \dots, l$ (i.e. the solid under consideration is supported at l points $M_v(x_v, y_v, z_v), v = 1, \dots, l$), condition 1 leads to $3 \times l$ linear equations where the A_{ijk}'s, the B_{ijk}'s and the C_{ijk}'s are the unknowns

$$\begin{cases} \hat{u}_x^* = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{o} A_{ijk}\left(\frac{x_v}{L}\right)^i\left(\frac{y_v}{H}\right)^j\left(\frac{z_v}{K}\right)^k = 0 \quad \forall v = 1, \dots, l \\ \hat{u}_y^* = \sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r} B_{ijk}\left(\frac{x_v}{L}\right)^i\left(\frac{y_v}{H}\right)^j\left(\frac{z_v}{K}\right)^k = 0 \quad \forall v = 1, \dots, l \\ \hat{u}_z^* = \sum_{i=0}^{s}\sum_{j=0}^{t}\sum_{k=0}^{u} C_{ijk}\left(\frac{x_v}{L}\right)^i\left(\frac{y_v}{H}\right)^j\left(\frac{z_v}{K}\right)^k = 0 \quad \forall v = 1, \dots, l \end{cases} \tag{29}$$

Note that the above equations can be avoided by choosing directly the following expression for the virtual field

$$\begin{cases} \hat{u}_x^* = \prod_{v=1}^{l}(x - x_v)(y - y_v)(z - z_v) \times \left(\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{o} A_{ijk}\left(\frac{x}{L}\right)^i\left(\frac{y}{H}\right)^j\left(\frac{z}{K}\right)^k\right) \\ \hat{u}_y^* = \prod_{v=1}^{l}(x - x_v)(y - y_v)(z - z_v) \times \left(\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r} B_{ijk}\left(\frac{x}{L}\right)^i\left(\frac{y}{H}\right)^j\left(\frac{z}{K}\right)^k\right) \\ \hat{u}_z^* = \prod_{v=1}^{l}(x - x_v)(y - y_v)(z - z_v) \times \left(\sum_{i=0}^{s}\sum_{j=0}^{t}\sum_{k=0}^{u} C_{ijk}\left(\frac{x}{L}\right)^i\left(\frac{y}{H}\right)^j\left(\frac{z}{K}\right)^k\right) \end{cases} \tag{30}$$

The counterpart is the fact that the derivation of the virtual field to get the virtual strain components involved in Eq. (1) will be somewhat more complicated. This approach will be limited in practice to the cases where the number of supports l is low.

4.2.2. The solid is clamped along a surface

Let us now assume that S_u is a curve described by the following equation

$$g(x,y,z) = 0 \quad \forall M(x,y,z) \in S_u \tag{31}$$

In the same way as the preceding case, the virtual field can be chosen as

$$\begin{cases} \hat{u}_x^* = g(x,y,z) \times \left(\sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{o} A_{ijk} \left(\frac{x}{L}\right)^i \left(\frac{y}{H}\right)^j \left(\frac{z}{K}\right)^k \right) \\ \hat{u}_y^* = g(x,y,z) \times \left(\sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{r} B_{ijk} \left(\frac{x}{L}\right)^i \left(\frac{y}{H}\right)^j \left(\frac{z}{K}\right)^k \right) \\ \hat{u}_z^* = g(x,y,z) \times \left(\sum_{i=0}^{s} \sum_{j=0}^{t} \sum_{k=0}^{u} C_{ijk} \left(\frac{x}{L}\right)^i \left(\frac{y}{H}\right)^j \left(\frac{z}{K}\right)^k \right) \end{cases} \tag{32}$$

to eliminate the equations coming from condition 1 since the special virtual field in Eq. (32) is directly admissible. The types of fields (28), (30) or (32) are chosen according to the shape of the solid and to the type of boundary conditions.

4.3. Equations induced by condition 2

Let us now examine condition 2. In practice, condition 2 leads to linear equations where the A_{ijk}'s, the B_{ijkl}'s and the C_{ijkl}'s are the unknowns. There are as many equations as unknown coefficients Q_{ij} to be identified, as can be easily verified in Eq. (7). These equations cannot be written in the general case since they depend on the expression of the special virtual fields in Eqs. (28), (30) or (32). The right-hand side of those equations is either 0 or 1, according to the parameter which is to be determined.

4.4. Equations induced by conditions 3, 4 and 5

One can easily check (see Grédiac et al. (2002)) that conditions 3, 4 and 5 lead in practice to linear equations between the unknown coefficients since they only involve conditions on the displacement field. Such equations are different from one case to the other. They are therefore not detailed here.

4.5. Final system

Eventually, since the equations are all linear, they lead to the following linear system

$$\mathbf{DY} = \mathbf{E} \tag{33}$$

where $\mathbf{D}$ is a rectangular matrix with as many columns as unknown coefficients A_{ijk}'s, B_{ijk}'s and C_{ijk}'s. Its number of rows depends on the number of parameters to be identified and on the number of conditions which must be verified since conditions 4 and 5 are optional, $\mathbf{E}$ is a vector which components are equal to 0 or 1 and $\mathbf{Y}$ is a vector which components are the unknowns coefficients A_{ijk}'s, B_{ijk}'s and C_{ijk}'s

$$\mathbf{Y}: \begin{bmatrix} A_{000} \\ A_{001} \\ \cdot \\ A_{00o} \\ A_{010} \\ \cdot \\ A_{mno} \\ B_{000} \\ B_{001} \\ \cdot \\ B_{00r} \\ B_{010} \\ \cdot \\ B_{pqr} \\ C_{000} \\ C_{001} \\ \cdot \\ C_{00u} \\ C_{010} \\ \cdot \\ C_{stu} \end{bmatrix} \tag{34}$$

4.6. Strategy for finding the coefficients characterizing the virtual fields

The number of columns in **D** is at least equal to the number of unknown parameters Q_{ij} to be identified (condition 2 above). Otherwise, **D** cannot be inverted. The actual number of columns is equal to the number of unknown coefficients A_{ijk}'s, B_{ijk}'s and C_{ijk}'s. It depends in practice on the choice of the maximum degree chosen for the expansion of the virtual fields. This number of columns, denoted n_{unk}, is equal to

$$n_{\text{unk}} = (n+1)(m+1)(o+1) + (p+1)(q+1)(r+1) + (s+1)(t+1)(u+1) \tag{35}$$

To determine exactly these unknowns, the number of linear equations available in the final system, denoted n_{equ} (which is also equal to the number of columns in **D**), must be equal to n_{unk} and the determinant of **D** must be different from 0. When the number of unknowns n_{unk} is greater than the number of equations n_{equ}, matrix **D** is rectangular and the number of solutions is a priori infinite. They are found by setting $(n_{\text{unk}} - n_{\text{equ}})$ coefficients A_{ijk}'s, B_{ijk}'s or C_{ijk}'s to values fixed arbitrarily. The linear system (33) is the rearranged and solved to obtain the remaining parameters (those which were not fixed a priori) if the determinant of the remaining square matrix of this new linear system is not zero. The value of this determinant cannot be discussed in the general case and the problem can be solved in practice by examining any possible combination of n_{equ} columns among the n_{unk} columns, as shown in Grédiac et al. (2002). The value of the A_{ijk}'s, B_{ijk}'s and C_{ijk}'s fixed a priori can be chosen randomly. In practice however, these values are set to zero (Grédiac et al., 2002). The determinant of the square matrix of the final system is computed in each case. If this determinant is greater than a selected threshold value, the remaining parameters are computed and the special virtual field is completely determined (by the $n_{\text{unk}} - n_{\text{equ}}$ parameters fixed a priori and by the n_{equ} parameters found by inversion of the system). In practice, with such a method, we will get a very large number of identified parameters: one per different special virtual field. Hence a very large number of estimates for each unknown parameter is available. This very interesting feature allows a possible selection of the "best" special virtual fields (and therefore the best estimate for an unknown parameter) with respect to a given criterion, for instance the stability when noisy data are processed (Grédiac et al., 2002).

4.7. Reduction of the dimension of the problem

The special virtual fields defined above directly provide the unknown parameters and a general numerical strategy has been proposed for finding them. The coefficients of the polynomials depend however on the actual strain field *inside* the solid to be characterized. Such quantities cannot be measured in practice whereas strain field over a surface can be measured using whole-field optical techniques. The dimension of the problem must be reduced from 3 to 2 for this reason and the solid becomes a plate. In practice, the strain field onto the external surface of the plate is measured and the strain inside it is deduced with an assumption. Two main cases can be distinguished in practice:

- In-plane problems: The through-thickness strain field is assumed to be equal to the strain field onto the external surface of the plate. Such a case is addressed in Grédiac et al. (2002).
- Bending problems: The through-thickness strain field is obtained with the Love–Kirchhoff assumption for instance.

5. Conclusion

The main features of the VFM used with special virtual fields are described in this paper. It has been shown that the so-called *special virtual fields* allow the direct identification of material constants from measured actual displacement/strain fields, provided that the constitutive law may be written as a polynomial of the strain components. Each unknown parameter is directly obtained with the virtual work of the applied loading produced with its associated special virtual fields. Hence, the iterative calculations carried out when finite element models are updated are avoided. A general numerical procedure for finding these special virtual fields has been given. The practical determination of these virtual fields, the accuracy and the stability of this general procedure can be assessed through an example described in a companion paper (Grédiac et al., 2002).

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